Reply to ''Comment on 'Model kinetic equation for low-density granular flow' ''

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The use of simple relaxation kinetic models for granular media is defended [see preceding Comment by Goldshtein and Shapiro, Phys. Rev. E **57**, 6210 (1998)]. [S1063-651X(98)11004-8]

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A molecular gas at low density is well described by the Boltzmann equation, but its complexity prohibits a transparent characterization of its solutions or their properties. Historically, kinetic models have been used to provide access to such solutions and their context. Model kinetic equations are obtained by replacing the Boltzmann operator $J[f,f]$ with a simpler, more tractable form that preserves its most important features (e.g., conservation laws). Within such constraints there is the flexibility to emphasize simplicity or accuracy, depending on the objectives of its use. Most knowledge of transport outside linear response and for boundary driven states has been obtained in this way. In [1] we extended this approach of kinetic modelling to explore the nature of solutions to the Boltzmann equation for inelastic collisions. Our objective was to address fundamental questions associated with the derivation of fluid dynamics from a more fundamental basis in a kinetic theory. Such questions arise because the energy is no longer a conserved hydrodynamic field, and the reference state about which spatial variations are measured is not local equilibrium, but rather an unknown time dependent cooling state (the homogeneous cooling state HCS). In short, we addressed the question of how inelasticity affects the derivation, form, and validity of fluid dynamics equations. From the chosen kinetic equation an exact solution was obtained for the HCS state, and the exact solution for small inhomogeneities was obtained through first order in the spatial gradients. The characterization in the Comment on our recent paper [1] by Goldshtein and Shapiro [2] of our treatment as ''incorrect'' and ''inconsistent'' certainly cannot apply to this analysis. Instead, they question the general validity of using a relaxation kinetic model for granular flow. Their primary basis for this position is that the kinetic model used in Ref. [1] predicted a homogeneous solution with a divergent fourth moment, while a calculation based on the Boltzmann equation indicates it is finite. In the following, we show below that this feature of the Boltzmann equation is reproduced by the kinetic model equation if the two adjustable constants are chosen to match the corresponding viscosity and cooling rate for the Boltzmann equation. Consequently, the only substantive argument against the kinetic model is removed.

Our chosen model kinetic equation for the distribution function $f(\mathbf{r},\mathbf{v},t)$ has the form

$$(\partial_t+\mathbf{v}\cdot\nabla)f(\mathbf{r},\mathbf{v},t)=J[f,f]\rightarrow-\zeta[f(\mathbf{r},\mathbf{v},t)-f_0(\mathbf{r},\mathbf{v},t|f)], \tag{1}$$

where the right side is an approximate representation of the Boltzmann collision operator for competition between scattering into and out of the velocity state, $\mathbf{v}$. It depends on two free quantities: an effective collision frequency ζ and a function $f_0(\mathbf{r},\mathbf{v},t|f)$, which is a functional of the distribution function through the constraints that the model kinetic equation yield exactly the same balance equations as the Boltzmann equation:

$$\int d\mathbf{v}\begin{pmatrix}1\\ \mathbf{v}\\ \frac{1}{2}mv^2\end{pmatrix}\zeta[f(\mathbf{r},\mathbf{v},t)-f_0(\mathbf{r},\mathbf{v},t|f)]=\begin{pmatrix}0\\ \mathbf{0}\\ (1-\epsilon^2)(w/\zeta)\end{pmatrix}. \tag{2}$$

The term on the right side of this equation proportional to $(1-\epsilon^2)$ represents the energy loss due to the inelastic collisions, where ϵ is the restitution coefficient and w is a bilinear functional known from the Boltzmann equation. A primary effect of inelastic collisions is the violation of detailed balance, implying that there is no longer an evolution toward a local Maxwellian. This violation of detailed balance is *assured* for the kinetic model by constraint (2), and the model kinetic equation agrees *exactly* with the Boltzmann equation in the subspace spanned by $(1,\mathbf{v},v^2)$. This defines a class of model kinetic equations, since the constraints do not uniquely determine $f_0(\mathbf{r},\mathbf{v},t|f)$. The choice in Ref. [1] is a Gaussian with parameters determined by Eq. (2),

$$f_0(\mathbf{r},\mathbf{v},t|f)=n(\mathbf{r},t)\left(\frac{m}{2\pi k_BT(\mathbf{r},t)\Delta}\right)^{3/2}\times e^{-m[\mathbf{v}-\mathbf{u}(\mathbf{r},t)]/2k_BT(\mathbf{r},t)\Delta}, \tag{3}$$

$$\Delta\equiv1-c(1-\epsilon^2),\quad c=\frac{2w}{3nk_BT\zeta}. \tag{4}$$

The functions $n(\mathbf{r},t)$, $T(\mathbf{r},t)$, and $\mathbf{u}(\mathbf{r},t)$ are the local density, temperature, and flow velocity which are defined as for a normal gas via moments of f. The appearance of the factor

Δ is due to constraint (2), and represents the violation of detailed balance. The extent of the violation is measured by the ratio of the mean free time to the cooling time, $c(1-\epsilon^2)$, where c is a numerical constant. As expected, no stationary solution to $f-f_0(\mathbf{r},\mathbf{v},t|f)=0$ exists except for $\epsilon=1$. However, a time dependent homogeneous scaling solution does exist and was determined exactly in Ref. [1] using $c=1$ for simplicity (see the Comment following Eq. (A13) in Ref. [1]). If, instead, w is estimated from the Boltzmann equation for local equilibrium and ζ is determined from the Boltzmann equation viscosity, then a value of $c=\frac{5}{12}$ is obtained. This makes no change in the results of [1] except that the dependence on the restitution coefficient is now obtained by replacing $1-\epsilon^2$ by $\frac{5}{12}(1-\epsilon^2)$ everywhere. The fourth moment of the distribution function is then finite for all ϵ, and very close to the value from the Boltzmann equation.

While this removes the specific concern of Ref. [2], it is possible to insist that problems remain since moments of higher degree still diverge. This is in fact the case, with the divergence occurring for degree $n\geqslant 2/[c(1-\epsilon^2)]$, a signature that the distribution function has an enhanced probability (relative to a Maxwellian) for large velocities. However, for a reasonable value of dissipation, $\epsilon=0.9$, the smallest moment to diverge is for $n=26$, which is clearly not relevant for the physical properties of interest (particularly hydrodynamics, which are equations for moments of degree $\leqslant 2$). More detailed analysis shows that the distribution function remains close to Maxwellian in this case for velocities up to five times the half-width, at which the value of f is $\sim 10^{-11}$. The enhanced probability occurs only for unphysical values of the velocity, having no effect on practical applications of the model. The reason for this feature of the kinetic model is easily traced to collapse of the Boltzmann spectrum for very rapidly decaying modes to the single common value ζ in the kinetic model. This can be corrected, e.g., by a velocity dependent collision frequency, but at the price of complicating the kinetic model considerably for limited gain.

In summary, we maintain that simple kinetic models for granular media are both physically acceptable and useful in practice. For example, their recent application to shear flow far from equilibrium has shown excellent agreement with simulation of the Boltzmann equation for rheological properties, and good agreement for the distribution function as well [3]. Theoretical analysis of such complex states based on the Boltzmann equation has not been possible to date. Kinetic models provide direct access to otherwise difficult or intractable problems, and therefore justify reasonable compromises in detailed accuracy.

[1] J. J. Brey, F. Moreno, and J. W. Dufty, Phys. Rev. E **54**, 445 (1996).

[2] A. Goldshtein and M. Shapiro, preceding paper, Phys. Rev. E **57**, 6210 (1998).

[3] J. J. Brey, M. J. Ruiz-Montero, and F. Moreno, Phys. Rev. E **55**, 1 (1997).